* First order ordinary differential equations:→

 # Differential Equation :- An equation involving independent variable x, dependent variable y and the differential coefficients $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots$ is called differential equation.

Let $y^2 = 4ax$

Differentiate w.r.to $x \Rightarrow 2y\frac{dy}{dx} = 4a$

Differentiate w.r.to $x \Rightarrow 2\left[y\frac{d^2y}{dx^2} + \frac{dy}{dx} \cdot \frac{dy}{dx}\right] = 0$

 # Order of a differential Equation:-

The order of a differential equation is the order of the highest derivative occurring in the differential equation. for example,

(i) $\frac{dy}{dx} = 1 + x + y + xy$ order = 1.

(ii) $\left(\frac{d^4y}{dx^4}\right)^3 - 4\frac{dy}{dx} + 4y = 5x\cos 3x$ order = 4.

 # Degree of a differential Equation :-

The degree of a differential equation is the degree of the highest order derivative, when differential coefficient are made free from radicals and fractions. In other words, the degree of a differential equation is the power of the highest order derivative occurring in differential equation when it is written as a polynomial in differential coefficients.

Ex:- (i) $\frac{dy}{dx} = 1+x+y+x$ Degree = 1.

(ii) $\left(\frac{d^4y}{dx^4}\right)^3 - 4\frac{dy}{dx} + 4y = 5x\cos 3x$, Degree = 3.

Differential Equation:-

(i) Variable separable form

(ii) Homogeneous Equations. (~~Degree~~ Power same)

(iii) Exact D.E.

(iv) Linear D.f.E. ($\frac{dy}{dx} + Py = Q$)

(v) Bernouilli's Equation.

Homogeneous Equation /D.E

① $(x^3+y^3)\frac{dy}{dx} = x^2y^1$ (Power should be same)

→ Put $y = vx$ $\therefore \frac{dy}{dx} = v + x\frac{dv}{dx}$.

Non - Homogeneous D.E :-

① $\frac{dy}{dx} = \frac{2x+4y+1}{3x-2y+3}$ Terms separate.

$\Rightarrow (3x-2y+3)\,dy = (2x+4y+1)\,dx$ (Powers are not same)

Exact Differential Equation:-

If M and N are function of x & y, the equation $Mdx + Ndy = 0$ is called exact diff. eq^n when there exists a function $f(x,y)$ of x and y such that

$d[f(x,y)] = Mdx + Ndy$ i.e, $\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = Mdx + Ndy$

where $\frac{\partial f}{\partial x}$ = Partial derivative of $f(x,y)$ with respect to y (treating x as constant)

The necessary and sufficient condition for the differential condition $Mdx + Ndy = 0$ to be exact is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

Working Rule for solving an exact differential equation:

step (i) Compare the given equation with $Mdx + Ndy = 0$ and find out M and N. Then find out $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$. If $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the given equation is exact.

step-(ii) Integrate M with respect to x treating y as a constant.

step-(iii) Integrate N with respect to y treating x as constant and omit those terms which have been already obtained by integrating M.

step-(iv) On adding the terms obtained in steps (ii) and (iii) and equating to an arbitrary constant, we get the required solution.

In other words, sol^n of an exact diff. eq^n is

$$\underset{\text{Regarding } y \text{ as constant}}{\int M dx} + \underset{\text{only those terms not containing } x.}{\int N dy} = C$$

Example-1 The solution of the differential equation

$$\frac{dy}{dx} = \frac{y^2\cos x + \cos y}{x\sin y - 2y\sin x}; \; y\left(\frac{\pi}{2}\right) = 0 \text{ is}$$

Sol= The given is

$$(y^2\cos x + \cos y)dx - (x\sin y - 2y\sin x)\,dy = 0$$

$$\Rightarrow \frac{\partial M}{\partial y} = 2y\cos x - \sin x = \frac{\partial N}{\partial x}$$

The given equation is Exact.

$$\underbrace{\int M dx}_{\text{Regarding } y \text{ as constant}} + \underbrace{\int N dy}_{\text{only those terms not containing } x} = c$$

$$\Rightarrow \int (y^2\cos x + \cos y)\,dx + \int 0\,dy = c$$

$$\Rightarrow y^2\sin x + x\cos y = c.$$

Apply $y\left(\frac{\pi}{2}\right) = 0 \Rightarrow 0 + \frac{\pi}{2}\cos 0 = c \Rightarrow c = \frac{\pi}{2}$

$$\Rightarrow y^2\sin x + x\cos y = \frac{\pi}{2}.$$

Linear Differential equation of first order

Linear in y:→ The general form of a linear differential eqn of first order is

$$\frac{dy}{dx} + P(x)y = Q(x) \quad \text{—(i)}$$

for example:→ (i) $\frac{dy}{dx} + xy = x^3$, (ii) $x\frac{dy}{dx} + 2y = x^3$,

(iii) $\frac{dy}{dx} + 2y = \sin x$ etc. are linear differential equations in y.

Working Rule for solving a linear differential equation in y:-

step(i): write the differential equation in the form $\frac{dy}{dx} + P(x)y = Q(x)$ and obtain P and Q.

step(ii): find integrating factor (I.F) given by $I.F = e^{\int P dx}$.

step(iii): Multiply both sides of equation in step(i) by I.F.

step(iv): Integrate both sides of the equation obtained in step (iii) w.r to x to obtain

$$\boxed{y(I.F) = \int Q(I.F)\,dx + C}$$

This gives the required solution.

Linear in X:- The general form of a linear differential equation of first order is

$$\frac{dx}{dy} + P(y)x = Q(y) \quad \text{—(i)}$$

for example:- (i) $(1+y^2) + (x-\tan^{-1}y)\frac{dy}{dx} = 0$, (ii) $(2x-10y^3)\frac{dy}{dx} + y = 0$, (iii) $dx + x\,dy = e^{-y}\sec^2 y\,dy$ etc. are linear differential equations in x.

Working Rule for solving linear differential equation in x:-

step(i): write the differential equation in the form $\frac{dx}{dy} + P(y)x = Q(y)$ and obtain P and Q.

Step(ii) :- find I.F. by using I.F. $= e^{-\int P\,dy}$

Step(iii) :- Multiply both sides of the differential equation in step(i) by I.F.

Step(iv) :- Integrate both sides of the equation obtained in step(iii) w.r.t y to obtain the solution given by $\boxed{x\,(I.F.) = \int Q\,(I.F.)\,dy + C}$ where C is the constant of integration.

Example-1 :- find the integrating factor of equation

$$(x^2+1)\frac{dy}{dx} + 2xy = x^2-1$$

Sol :- The given equation is

$$\frac{dy}{dx} + \frac{2x}{x^2+1}y = \frac{x^2-1}{x^2+1}$$

comparing with $\frac{dy}{dx} + Py = Q$, $P = \frac{2x}{x^2+1}$

$$I.F. = e^{\int P\,dx} = e^{\int \frac{2x\,dx}{1+x^2}} = e^{\ln(1+x^2)} = \underline{1+x^2}$$

(2) The solution of the differential equation $(1+y^2) + (x - e^{\tan^{-1}y})\frac{dy}{dx} = 0$ is

Sol :- The given equation is

$$(x - e^{\tan^{-1}y})\frac{dy}{dx} = -(1+y^2)$$

equation

$$\frac{dx}{dy} = -\left(\frac{x - e^{\tan^{-1}y}}{1+y^2}\right) \Rightarrow \frac{dx}{dy} + \frac{1}{1+y^2}x = \frac{e^{\tan^{-1}y}}{1+y^2} \quad \text{--(i)}$$

This is a linear differential equation of the form $\frac{dx}{dy} + R(y)x = S(y)$

$$R = \frac{1}{1+y^2}, \quad S = \frac{e^{\tan^{-1}y}}{1+y^2}$$

Integrating factor $= e^{\int R\,dy} = e^{\int \frac{dy}{1+y^2}} = e^{\tan^{-1}y}$

Multiplying (i) by I.F. and integrating,

$$xe^{\tan^{-1}y} = \int \frac{e^{\tan^{-1}y}}{1+y^2} \cdot e^{\tan^{-1}y}\,dy$$

$$= \int \frac{(e^{\tan^{-1}y})^2\,dy}{1+y^2} = \frac{(e^{\tan^{-1}y})^2}{2} + \frac{k}{2}$$

$$\therefore \boxed{2xe^{\tan^{-1}y} = e^{2\tan^{-1}y} + k}$$

Some examples on Exact Differential Equation:-

(1) find the value of α, assuming that the differential equation

$(1+x^2y^3+\alpha x^2y^2)\,dx + (2+x^3y^2+x^3y)\,dy = 0$ is exact

Sol:- The given differential equation is

$$(1+x^2y^3+\alpha x^2y^2)dx + (2+x^3y^2+x^3y)dy = 0 \quad \text{--(1)}$$

comparing (1) with $Mdx + Ndy = 0$, we have

$$M = 1+x^2y^3+\alpha x^2y^2 \text{ and } N = 2+x^3y^2+x^3y.$$

$$\frac{\partial M}{\partial y} = 3x^2y^2 + 2\alpha x^2 y \text{ and } \frac{\partial N}{\partial x} = 3x^2y^2 + 3x^2y.$$

As the given differential equation is exact, so

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

$$\therefore \quad 3x^2y^2 + 2\alpha x^2 y = 3x^2y^2 + 3x^2y.$$

$$\Rightarrow 2\alpha = 3 \Rightarrow \alpha = \frac{3}{2}$$

(8) Solve the differential equation,

$$(x^4 - 2xy^2 + y^4)\,dx - (2x^2y - 4xy^3 + \sin y)\,dy = 0$$

Sol= The given differential equation is

$$(x^4 - 2xy^2 + y^4)\,dx - (2x^2y - 4xy^3 + \sin y)\,dy = 0 \quad (1)$$

Comparing equation (1) with $Mdx + Ndy = 0$, we observe that

$$M = x^4 - 2xy^2 + y^4, \quad N = (-2x^2y + 4xy^3 - \sin y)$$

$$\therefore \quad \frac{\partial M}{\partial y} = -4xy + 4y^3 \quad \text{and} \quad \frac{\partial N}{\partial x} = -4xy + 4y^3$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, therefore the given eq^n (1) is exact.

Hence, the solution is

$$\int_{y\ \text{constant}} M\,dx + \int (\text{terms in } N \text{ not containing } x)\,dy = c$$

$$\Rightarrow \int_{y\ \text{constant}} (x^4 - 2xy^2 + y^4)\,dx + \int -\sin y\,dy = c$$

$$\Rightarrow \frac{x^5}{5} - 2y^2 \cdot \frac{x^2}{2} + y^4 \cdot x + \cos y = c$$

$$\Rightarrow \frac{x^5}{5} - x^2y^2 + xy^4 + \cos y = c$$

Solve the differential equation:-

$$\frac{2x}{y^3}\,dx + \left(\frac{y^2-3x^2}{y^4}\right)dy = 0$$

Sol= The given differential equation is

$$\frac{2x}{y^3}\,dx + \left(\frac{y^2-3x^2}{y^4}\right)dy = 0 \quad -(1)$$

comparing equation (1) with $Mdx + Ndy = 0$, we observe that

$$M = \frac{2x}{y^3} \quad \text{and} \quad N = \frac{1}{y^2} - \frac{3x^2}{y^4}$$

$$\therefore \quad \frac{\partial M}{\partial y} = \frac{-6x}{y^4} \quad \text{and} \quad \frac{\partial N}{\partial x} = -\frac{6x}{y^4}$$

since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, therefore the given eq^n (1) is exact.

Hence, the solution is

$$\int_{y\,\text{constant}} M dx + \int (\text{terms in } N \text{ not containing } x)\,dy = c$$

i.e, $$\int_{y\,\text{constant}} \frac{2x}{y^3}\,dx + \int \left(\frac{1}{y^2}\right) dy = c$$

or $$\frac{2}{y^3}\int_{y\,\text{constant}} x\,dx + \int \frac{1}{y^2}\,dy = c$$

$$\Rightarrow \quad \frac{x^2}{y^3} - \frac{1}{y} = c$$

$$\Rightarrow \quad \boxed{x^2 - y^2 = cy^3.}$$

Scanned with CamScanner

Equation of first order and Higher degree :→

A differential equation $f\left(x, y, \frac{dy}{dx}\right)=0$, where degree $\frac{dy}{dx} \geq 1$ is said to be non-linear differential equation of first order an higher degree. It is generally of the type

$$P_0\left(\frac{dy}{dx}\right)^n + P_1\left(\frac{dy}{dx}\right)^{n-1} + P_2\left(\frac{dy}{dx}\right)^{n-2} + \dots + P_{n-1}\frac{dy}{dx} + P_n = 0.$$

where $P_0, P_1, P_2, \dots, P_{n-1}, P_n$ are functions of x and y. for convenience, we write $\frac{dy}{dx} = p$ and the above equation takes the form

$$P_0 p^n + P_1 p^{n-1} + P_2 p^{n-2} + \dots + P_{n-1} p + P_n = 0.$$

Equations Solvable for p :→

Suppose the equation

$p^n + A_1 p^{n-1} + A_2 p^{n-2} + \dots + A_{n-1} p + A_n = 0$ can be put in the form

$$[p - f_1(x,y)][p - f_2(x,y)] \dots [p - f_n(x,y)] = 0$$

To obtain the solution of given equation, equate each factor to zero, which gives equations of first order and first degree namely

$$p = f_1(x,y), \quad p = f_2(x,y), \dots, p = f_n(x,y)$$

$$\frac{dy}{dx} = f_1(x,y), \quad \frac{dy}{dx} = f_2(x,y), \dots, \frac{dy}{dx} = f_n(x,y).$$

(i) Inde

I

i)

Let their solution be

$$f_1(x, y, c_1) = 0$$
$$f_2(x, y, c_2) = 0$$
$$f_3(x, y, c_3) = 0$$
$$\cdots \quad \cdots$$
$$\cdots \quad \cdots$$
$$f_n(x, y, c_n) = 0$$

where $c_1, c_2, \dots, c_n$ are arbitrary constant. The most general solution of the original differential equation is

$$[f_1(x,y,c_1)].[f_2(x,y,c_2)].[f_3(x,y,c_3)] \cdots [f_n(x,y,c_n)] = 0.$$

Since the given equation is of first order therefore the general solution cannot have more than one arbitrary constant. Hence, there is no loss of generality if we take,

$$c_1 = c_2 = c_3 = \cdots = c_3 = c \text{ (say)}$$

Hence, the general solution of the given equation can be put as

$$[f_1(x,y,c)][f_2(x,y,c)] \cdots [f_n(x,y,c)] = 0.$$

Working Rule's

(1) Put $\frac{dy}{dx} = p$ in the given differential equation.

(2) Make R.H.S zero and factorize L.H.S into linear factors of p.

...ation is (i) Indefinite ...

(3) Equate each linear factor to zero and find ... solutions taking constant c same in all solutions.

(4) Multiply all the solutions obtained in the above step and equate to zero for obtaining general solution of the given differential equation.

Example-1 Solve the differential equation

$$\left(\frac{dy}{dx}\right)^2 - 5\frac{dy}{dx} + 6 = 0$$

Sol= The given equation is

$$\left(\frac{dy}{dx}\right)^2 - 5\frac{dy}{dx} + 6 = 0$$

Putting $\frac{dy}{dx} = p$, we get $p^2 - 5p + 6 = 0$

or $(p-3)(p-2) = 0$

$\therefore$ either $p - 3 = 0$

$\Rightarrow p = 3$

$\therefore \frac{dy}{dx} = 3$

Integrating,

$y = 3x + c$

or $y - 3x - c = 0$

or $p - 2 = 0$

$\Rightarrow p = 2$

$\therefore \frac{dy}{dx} = 2$

Integrating,

$y = 2x + c$

or $y - 2x - c = 0$.

Therefore, the most general solution is

$$(y-3x-c)(y-2x-c) = 0.$$

Ty to solve the differential equation $p^3+2xp^2-y^2p^2-2xy^2p=0$ 1.

(i= The given equation is

$$p^3+2xp^2-y^2p^2-2xy^2p=0$$

$$p(p^2+2xp-y^2p-2xy^2)=0$$

$$p(p+2x)(p-y^2)=0$$

$$\therefore\quad p=0,\ -2x,\ y^2$$

ie, $\frac{dy}{dx}=0$; $\frac{dy}{dx}=-2x$; $\frac{dy}{dx}=y^2$

Integrating, we have,

$$\int dy = c$$

$$\therefore\quad y=c$$

$$\Rightarrow y-c=0$$

$$\int dy = \int -2x\,dx + c$$

$$\therefore\ y=-x^2+c$$

$$\Rightarrow\ y+x^2-c=0$$

$$\int\frac{dy}{y^2}=\int dx+c$$

$$\therefore -\frac{1}{y}=x+c$$

$$\Rightarrow -1=xy+cy$$

$$\Rightarrow xy+cy+1=0$$

Hence, the most general solution is

$$(y-c)(y+x^2-c)(xy+cy+1)=0.$$

(3) Solve the differential equation

$$x^2\left(\frac{dy}{dx}\right)^2+xy\frac{dy}{dx}-6y^2=0.$$

Sol= The given equation is

$$x^2\left(\frac{dy}{dx}\right)^2+xy\frac{dy}{dx}-6y^2=0.$$

Putting $\frac{dy}{dx} = p$, we get $x^2p^2 + xyp - 6y^2 = 0$

$$\Rightarrow (px+3y)(px-2y) = 0$$

$px + 3y = 0$ gives

$px = -3y$

$\frac{dy}{dx} x = -3y$

$\frac{dy}{y} = -3 \frac{dx}{x}$

Integrating,

$\log y = -3\log x + c,$

$= \log x^{-3} + \log c$

$= \log x^{-3} \cdot c$

$\therefore \quad y = \frac{c}{x^3}$

$yx^3 = c$

$yx^3 - c = 0$

$px - 2y = 0$ gives

$px = 2y$

$\frac{dy}{dx} x = 2y$

$\frac{dy}{y} = 2 \frac{dx}{x}$

Integrating,

$\log y = 2\log x + \log c$

$= \log x^2 + \log c$

$= \log x^2 c$

$\therefore \quad y = x^2 c$

$\frac{y}{x^2} = c$

$\Rightarrow \left(\frac{y}{x^2} - c\right) = 0.$

Hence, the required solution is

$$(yx^3 - c)\left(\frac{y}{x^2} - c\right) = 0.$$

Scanned with CamScanner

II Equation Solvable for y :-

If y can be expressed explicitly in terms of x and p, then the equation is solvable for y.

consider $\quad y = f(x,p) \quad -(1)$

Differentiating w.r.to x, we get

$$\frac{dy}{dx} = f\left(x, p, \frac{dp}{dx}\right) \quad -(2)$$

$$p = f\left(x, p, \frac{dp}{dx}\right)$$

The equation is in two variables x and p and can be solved. Let its solution be

$$\phi(x, p, c) = 0 \quad -(3)$$

Eliminating p b/w (1) and (3), we will get a relation between x, y and c which will be the required solution.

Working Rule :

(1) Express the given equation in the form $y = f(x,p)$.

(2) Differentiate w.r to x and replace $\frac{dy}{dx}$ by p so that the equation has two variables p and x.

(3) Solve the equation obtained in step 2.

(4) Let the solution of equation be $\phi(x,p,c) = 0$.

(5) Eliminate p from equations of steps 1 and 4.

Example-1 Solve the differential equation $y = 2px + p^4x^2$ and calculate

Sol:- The given equation is $y = 2px + p^4x^2$ —(1)

Differentiating w.r.to x, we have,

$$\frac{dy}{dx} = 2p + 2x\frac{dp}{dx} + 2xp^4 + 4p^3x^2\frac{dp}{dx}$$

$$p = 2p + 2x\frac{dp}{dx} + 2xp^4 + 4p^3x^2\frac{dp}{dx}$$

$$\Rightarrow p + 2x\frac{dp}{dx} + 2xp^4 + 4p^3x^2\frac{dp}{dx} = 0$$

$$\Rightarrow \left(p + 2x\frac{dp}{dx}\right)(1 + 2p^3x) = 0$$

The factor $1 + 2p^3x = 0$ is discarded as it does not contain $\frac{dp}{dx}$.

Thus, $p + 2x\frac{dp}{dx} = 0$, $2x\frac{dp}{dx} = -p$

or $\frac{2dp}{p} = -\frac{dx}{x}$ $\frac{2dp}{p} = -\frac{dx}{x}$

Integrating, $2\log p = -\log x + \log c$

$$\log p^2 = \log\frac{c}{x}$$

$$p^2 = \frac{c}{x} \Rightarrow p = \sqrt{\frac{c}{x}}.$$

Putting this value of p in (1), we have

$$y = 2\sqrt{\frac{c}{x}}\,x + \frac{c^2}{x^2}\cdot x^2$$

$$y = 2\sqrt{cx} + c^2$$

which is the complete solution.

...ve the differential equation $y = 3x + \log p$.

... The given equation is $y = 3x + \log p$ —(1)

Differentiating w.r.to x, we have

$$\frac{dy}{dx} = 3 + \frac{1}{p}\frac{dp}{dx}$$

$$p = 3 + \frac{1}{p}\frac{dp}{dx}$$

$$dx = \frac{dp}{p(p-3)}$$

$$dx = \frac{1}{3}\left[\frac{1}{p-3} - \frac{1}{p}\right]dp$$

Integrating, $x = \frac{1}{3}[\log(p-3) - \log p] + \log c_1$

$$3x = \log\frac{p-3}{p} + \log c_2$$

$$3x = \log\frac{p-3}{p} + \log c_2$$

$$3x = \log\left(\frac{p-3}{p}\right)\cdot c_2$$

$$e^{3x} = \frac{p-3}{p}\cdot c_2 \Rightarrow \frac{p-3}{p} = ce^{3x}$$

$$1 - \frac{3}{p} = ce^{3x} \Rightarrow p = \frac{3}{1-ce^{3x}}$$

Putting this value of p in (1), we have

$$y = 3x + \log\frac{3}{1-ce^{3x}}$$

which is the required solution.

(3) Solve the differential equation
$$p^3+p=e^y.$$

Sol = The given equation is $p^3+p=e^y$

Taking log of both sides, we have

$$\log p(p^2+1) = \log e^y$$

$$\log p+\log(p^2+1)=y$$

$$y=\log p+\log(1+p^2)$$

Differentiating w.r.to x, we have,

$$\frac{dy}{dx}=\frac{1}{p}\frac{dp}{dx}+\frac{2p}{1+p^2}\frac{dp}{dx}$$

$$p=\frac{1}{p}\frac{dp}{dx}+\frac{2p}{1+p^2}\frac{dp}{dx}$$

$$dx=\left(\frac{1}{p^2}+\frac{2}{1+p^2}\right)dp$$

Integrating,

$$x=\frac{-1}{p}+2\tan^{-1}p+c. \quad -(1)$$

Also, $y=\log p+\log(1+p^2)$ —(2)

These two equations (1) and (2) form the required solution.

Equations solvable for x :-

The equation is said to be solvable for x if x can be expressed explicitly in terms of y and p. Such an equation can be put in the form

$$x = f(y, p) \quad -(1)$$

Differentiating, (1) w.r.to y, we get

$$\frac{dx}{dy} = f\left(y, p, \frac{dp}{dy}\right)$$

$$\frac{1}{p} = f\left(y, p, \frac{dp}{dy}\right) \quad -(2)$$

The equation involves two variables y and p.

Suppose the solution is

$$\phi(y, p, c) = 0 \quad -(3)$$

where c is any arbitrary constant.

Eliminating p from (1) and (3), we get the solution of the equation (1). If elimination of p is not possible, then values of x and y expressed in terms of parameters p together form the solution of the equation.

Solve the differential equation $x = y + p^2$.

Ex:-(1) Solve the differential equation $x = y + p^2$.

Sol :- The given equation is $x = y + p^2 \quad -(1)$

Diff. w.r.to y, we get

$$\frac{dx}{dy} = 1 + 2p\frac{dp}{dy}.$$

$$\Rightarrow \frac{1}{p} = 1 + 2p\frac{dp}{dy} \Rightarrow 2p\frac{dp}{dy} = \frac{1}{p} - 1$$

$$\Rightarrow \frac{dp}{dy} = \frac{1-p}{2p^2} \Rightarrow \frac{2p^2}{1-p}\,dp = dy$$

$$\Rightarrow -2\left[p+1+\frac{1}{p-1}\right]dp = dy$$

Integrating, we get $c - 2\left[\frac{p^2}{2} + p + \log(p-1)\right] = y.$

$$y = c - [p^2 + 2p + 2\log(p-1)].$$

Putting the value of y in (1), we get

$$x = c - [2p + 2\log(p-1)].$$

which is the required solution.

(2) Solve the differential equation $x = y + a\log p$.

Sol: The given equation is $x = y + a\log p$. —(1)

Differentiating w.r.to y, we get

$$\frac{dx}{dy} = 1 + \frac{a}{p}\frac{dp}{dy} \Rightarrow \frac{1}{p} = 1 - \frac{a}{p}\frac{dp}{dy}$$

$$\Rightarrow (1-p) = a\frac{dp}{dy} \Rightarrow dy = a\frac{dp}{1-p}$$

Integrating, we get

$$y = c - a\log(1-p) \qquad \text{—(2)}$$

Equation (1) & (2) constitute the required solution.

Scanned with CamScanner

CLAIRAUT'S EQUATION :-

The differential equation $y = px + f(p)$ (Clairaut's eqⁿ)

To solve the differential equation $\boxed{y = px + f(p)}$

The given equation is $y = px + f(p)$ —(1)

Differentiating w.r.t x, we get

$$\frac{dy}{dx} = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx}$$

$$p = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx}$$

$$[x + f'(p)]\frac{dp}{dx} = 0$$

Neglecting $x + f'(p)$, we get $\frac{dp}{dx} = 0$

Integrating both sides, we obtain

$$p = c$$

Putting $p = c$ in (1), we get

$y = cx + f(c)$, which is the required solution

Examples

(1) Solve the differential equation $(y - px)^2 = 1 + p^2$

Sol :- The given equation is

$$(y - px)^2 = 1 + p^2$$

$$y - px = \sqrt{1 + p^2}$$

$$y = px + \sqrt{1 + p^2} \quad -(1)$$

which is a clairaut's equation of the form

$y = px + f(p)$

Replacing p by Constant c in (1), we get

$y = cx + \sqrt{1+c^2}$, which is required solution

(2) Solve the differential equation

$\sin px \cos y = \cos px \sin y + p$

Sol:- The given differential equation is

$$\sin px \cos y = \cos px \sin y + p$$

$$\sin px \cos y - \cos px \sin y = p$$

$$\sin(px - y) = p$$

$$px - y = \sin^{-1} p$$

$$y = px - \sin^{-1} p \qquad -(1)$$

Which is a clairaut equation of the form

$y = px + f(p)$

Replacing p by constant c in (1), we get

$$y = cx - \sin^{-1} c.$$

Which is required solution.